Algebraic Topology: Midterm exam
March 5, 2004

Attempt all six questions. Each question is worth 20 points. The maximum score is 120

1. Let $(G, 1)$ be a path-connected topological group. Let $\alpha, \beta : [0,1] \to G$ be loops based at 1.

   (a) Show that $\alpha * \beta$ is homotopic to the loop $t \mapsto \alpha(t)\beta(t)$ and also the loop $t \mapsto \beta(t)\alpha(t)$

   (b) Conclude that $\pi_1(G, 1)$ is abelian.

2. Let X denote the wedge of two circles.

   (a) Construct a connected cover of X of order 3 that is not Galois and show that it is not Galois.

   (b) Show that if Y is a connected cover of X of order 3 that is not Galois, then the only deck transformation of Y is the identity.

3. Let G be a free group on n generators and let H be a subgroup of G of index k. Show that H is a free group on $kn - k + 1$ generators. You may use the fact that if a tree T has V vertices and E edges, then $V - E = 1$.

4. Let X be the Möbius band, i.e., the quotient of $[0,1] \times [0,1]$ by the equivalence relation $(0, y) \sim (1, 1-y), y \in [0,1]$. Show that there is no retraction from X to its boundary (the boundary of X is the image of $[0,1] \times \{0,1\} \subset [0,1] \times [0,1]$ under the quotient map).

5. Let $\tilde{X}$ be a connected, locally path-connected space that covers X with $\pi_1(\tilde{X}) = 1$. Show that if Y is a connected cover of X, then $\tilde{X}$ is a cover of Y.

6. Let $Y \subset \mathbb{R}^2$ be the closure of the graph of the function $f(x) = sin(\pi/x), x \in (0,1)$. Let X be the space obtained from $Y \coprod [0,1]$ by identifying the points $(0,0)$ and $(1,0)$ in Y with the endpoints 0 and 1 of the interval $[0,1]$. Let $\tilde{X}$ be the closure of the graph of the function $f : \mathbb{R} - \mathbb{Z} \to \mathbb{R}$, $f(x) = \sin(\frac{\pi}{x - [x]})$. Show that if Y is a *connected* cover of X, then $\tilde{X}$ is a cover of Y.